

TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
SOLUTION SKETCHES TO HOMEWORK 12

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Homework 12.1 (High-dimensional Schwarz lemma*). Let $f: B_1(0) \rightarrow \mathbb{C}$ be holomorphic such that $f(0) = 0$, where $B_1(0)$ denotes the unit ball in $\mathbb{C}^n$, where $n \geq 2$. Assume there exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in B_1(0)$.

- Show $|f(z)| \leq M\|z\|$ for every $z \in B_1(0)$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{C}^n$.
- If $n = 1$, the equality $|f(z)| = M|z|$ for some nonzero $z \in B_1(0)$ implies $f(z) = Maz$ for some $a \in \partial B_1(0)$. In particular, when $M > 0$ then f is biholomorphic. Show that for $n \geq 2$ there is a holomorphic function $f: B_1(0) \rightarrow B_1(0)$ with $f(0) = 0$ and $\|f(z)\| = \|z\|$ for some $z \in B_1(0) \setminus \{0\}$ that is not even injective¹.

Homework 12.2 (A stronger version of the identity theorem). Let $D \subset \mathbb{C}^n$ be a domain and $f: D \rightarrow \mathbb{C}$ be holomorphic. Assume there exists $a \in D$ such that for every multiindex $\alpha \in \mathbb{N}_0^n$, we have $D^\alpha f(a) = 0$. Show $f = 0$.

Solution. By Corollary 8.6, we can write f locally near a as a convergent power series, i.e. there exists $r > 0$ such that for all $z \in B_r(a)$ it holds that

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{D^\alpha f(a)}{\alpha!} (z - a)^\alpha.$$

By assumption, f vanishes identically on $B_r(a)$. By the identity theorem (Corollary 8.4) we conclude $f = 0$.

Homework 12.3 (Consequences of Hartogs' extension theorem). Let $n \geq 2$ and $U \subset \mathbb{C}^n$ be open. Show the following statements

- If $f: U \setminus \{a\} \rightarrow \mathbb{C}$ is holomorphic for a given point $a \in \mathbb{C}^n$, then f can be extended to a holomorphic function $f: U \rightarrow \mathbb{C}$.
- If $K \subset \mathbb{C}^n$ is compact and such that $\mathbb{C}^n \setminus K$ is connected, then every holomorphic function $f: \mathbb{C}^n \setminus K \rightarrow \mathbb{C}$ can be extended to an entire function.
- If $f: U \rightarrow \mathbb{C}$ is holomorphic, then f cannot have an isolated zero.
- If $f: \mathbb{C}^n \rightarrow \mathbb{C}$ is entire, then $\{f = 0\}$ is either empty or unbounded.

Solution. In order to get an idea how to apply the version of Hartogs' extension theorem proven in the lecture for a. and b., it helps to draw a picture, having in mind the case $n = 2$.

a. It is enough to consider the case $a = 0$ and $U = D_R^n(0)$ for some polydisc with equal radii $R > 0$. We choose $D = D_R^{n-1}(0)$ and $A(0, R)$ in Theorem 8.9. Then for any $z \in D_R^{n-1}(0) \setminus \{0\}$ the function f is holomorphic on a set of the form neighborhood of $B_\varepsilon(z) \times B_R(0)$, where $\varepsilon > 0$ might depend on z . Thus, we can extend it in a holomorphic way to $D_R^n(0)$, which proves the claim.

b. Since $K \subset \mathbb{C}^n$ is compact, there exists $R > 0$ such that $K \subset \bar{D}_R^n(0)$. We will prove that $f|_{\mathbb{C}^{n-1} \times \mathbb{C} \setminus \bar{D}_R^n(0)}$ can be extended to an entire function. By the identity theorem it then follows that this extension also extends $f: \mathbb{C}^n \setminus K \rightarrow \mathbb{C}$. (Here we use the assumption that

Date: December 16, 2024.

¹Cartan's uniqueness theorem states if $U \subset \mathbb{C}^n$ is bounded and $f: U \rightarrow U$ has a fixed point $a \in U$ with $Df(a) = \text{Id}$ then $f(z) = z$ for all $z \in U$.

$\mathbf{C}^n \setminus K$ is connected.) We apply Theorem 8.9 with $D = \mathbf{C}^{n-1}$ and $A(R, \infty) := \mathbf{C} \setminus \bar{D}_R^1(0)$. Note f is holomorphic on $D \times A(R, \infty)$. Moreover, for any $z \in \mathbf{C}^{n-1} \setminus D_{2R}^{n-1}(0)$ the function f is holomorphic on a neighborhood of $\{z\} \times \mathbf{C}$, so that by Theorem 8.9 it can be extended to a holomorphic function on $\mathbf{C}^{n-1} \times \mathbf{C} = \mathbf{C}^n$. This proves the claim.

c. If f has an isolated zero in $a \in U$, then there exists $r > 0$ such that $1/f$ is holomorphic on $B_r(a) \setminus \{a\}$. However, $1/f$ is unbounded in a neighborhood of a , so that it cannot be extended to a holomorphic function on $B_r(a)$. This contradicts a.

d. Assume $\{f = 0\}$ is non-empty. If this set is compact, there exists $R > 0$ such that the map $1/f$ is holomorphic on $\mathbf{C} \setminus \bar{B}_R(0)$. By b., it can be extended to an entire function. Call this function g . Then $f g = 1$ on an open set. Hence $f g = 1$ everywhere by the identity theorem. This contradicts the fact that f has a zero.

Homework 12.4 (Merry Christmas*). Enjoy your holidays!